

SEIDEL ELEMENTS AND MIRROR TRANSFORMATIONS FOR TORIC STACKS

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ABSTRACT. We give a precise relation between the mirror transformation and the Seidel elements for weak Fano toric Deligne-Mumford stacks. Our result generalizes the corresponding result for toric varieties proved by González and Iritani in [5].

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1. INTRODUCTION

In [5], González and Iritani gave a precise relation between the mirror map and the Seidel elements for a smooth projective weak Fano toric variety X . The goal of this paper is to generalize the main theorem of [5] to a smooth projective weak Fano toric Deligne-Mumford stack \mathcal{X} .

Let \mathcal{X} be a smooth projective weak Fano toric Deligne-Mumford stack, the mirror theorem can be stated as an equality between the I -function and the J -function via a change of coordinates, called mirror map (or mirror transformation). We refer to [3] and section 4.1 of [6] for further discussions.

Key words and phrases. Seidel elements, mirror transformations, Batyrev relations, Weak Fano, toric Deligne-Mumford stacks.

Let Y be a monotone symplectic manifold. For a loop λ in the group of Hamiltonian symplectomorphisms on Y , Seidel [10] constructed an invertible element $S(\lambda)$ in (small) quantum cohomology counting sections of the associated Hamiltonian Y -bundle $E_\lambda \rightarrow \mathbb{P}^1$. The Seidel element $S(\lambda)$ defines an element in $\text{Aut}(QH(Y))$ via quantum multiplication and the map $\lambda \mapsto S(\lambda)$ gives a representation of $\pi_1(\text{Ham}(Y))$ on $QH(Y)$. The construction was extended to all symplectic manifolds by McDuff and Tolman in [9]. Let D_1, \dots, D_m be the classes in $H^2(X)$ Poincaré dual to the toric divisors. When the loop λ is a circle action, McDuff and Tolman [9] considered the Seidel element \tilde{S}_j associated to an action λ_j that fixes the toric divisor D_j . The definition of Seidel representation and Seidel element were extended to symplectic orbifolds by Tseng-Wang in [11].

Given a circle action on X (resp. \mathcal{X}), the Seidel element in [5] (resp. [11]) is defined using the small quantum cohomology ring. In this paper, we need to define it, for smooth projective Deligne-Mumford stack, with deformed quantum cohomology to include the bulk deformations. For weak Fano toric Deligne-Mumford stack, the mirror theorem in [6] shows that the mirror map $\tau(y) \in H_{orb}^{\leq 2}(\mathcal{X})$, therefore, we will only need bulk deformations with $\tau \in H_{orb}^{\leq 2}(\mathcal{X})$.

We consider the Seidel element \tilde{S}_j associated to the toric divisor D_j as well as the Seidel element \tilde{S}_{m+j} corresponding to the box element s_j . The Seidel element in definition 2.2 shows that $S = q_0 \tilde{S}$ is a pull-back of a coefficient of the J -function $J_{\mathcal{E}_j}$ of the associated orbifold bundle \mathcal{E}_j , hence we can use the mirror theorem for \mathcal{E}_j to calculate \tilde{S}_j when \mathcal{E}_j is weak Fano.

We extend the definition of the Batyrev element \tilde{D}_j to weak Fano toric Deligne-Mumford stacks via partial derivatives of the mirror map $\tau(y)$. As analogues of the Seidel elements in B-model, the Batyrev elements can be explicitly computed from the I -function of \mathcal{X} . The following theorem states that the Seidel elements and the Batyrev elements only differ by a multiplication of a correction function.

Theorem 1.1. *Let X be a smooth projective toric Deligne-Mumford stack with $\rho^S \in cl(C^S(\mathcal{X}))$.*

(i) *the Seidel element \tilde{S}_j associated to the toric divisor D_j is given by*

$$\tilde{S}_j(\tau(y)) = \exp\left(-g_0^{(j)}(y)\right) \tilde{D}_j(y)$$

where $\tau(y)$ is the mirror map of \mathcal{X} and the function $g_0^{(j)}$ is given explicitly in (40);

(ii) *the Seidel element \tilde{S}_{m+j} corresponding to the box element s_j is given by*

$$\tilde{S}_{m+j}(\tau(y)) = \exp\left(-g_0^{(m+j)}\right) y^{-D_{m+j}^{SV}} \tilde{D}_{m+j}(y),$$

where $\tau(y)$ is the mirror map of \mathcal{X} and the function $g_0^{(m+j)}$ is given explicitly in (52).

It appears that the correction coefficients in the above theorem coincide with the instanton corrections in theorem 1.4 in [2]. This phenomenon also indicates the deformed quantum cohomology of the toric Deligne-Mumford stack \mathcal{X} is isomorphic to the Batyrev ring given in [6].

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2. SEIDEL ELEMENTS AND J -FUNCTIONS

2.1. Generalities. In this section, we will fix our notation and construct the Seidel elements of smooth projective Deligne-Mumford stacks using τ -deformed quantum cohomology.

Let \mathcal{X} be a smooth projective Deligne-Mumford stack, equipped with a \mathbb{C}^\times action.

Definition 2.1. The associated orbifold bundle of the \mathbb{C}^\times -action is the \mathcal{X} -bundle over \mathbb{P}^1

$$\mathcal{E} := \mathcal{X} \times (\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^\times \rightarrow \mathbb{P}^1,$$

where \mathbb{C}^\times acts on $\mathbb{C}^2 \setminus \{0\}$ via the standard diagonal action.

Let ϕ_1, \dots, ϕ_N be a basis for the orbifold cohomology ring $H_{orb}^*(\mathcal{X}) := H^*(\mathcal{IX}; \mathbb{Q})$ of \mathcal{X} , where \mathcal{IX} is the inertia stack of \mathcal{X} . Let ϕ^1, \dots, ϕ^N be the dual basis of ϕ_1, \dots, ϕ_N with respect to the orbifold Poincaré pairing. Furthermore, let $\hat{\phi}_1, \dots, \hat{\phi}_M$ denote a basis for the orbifold cohomology $H_{orb}^*(\mathcal{E}) := H^*(\mathcal{IE}; \mathbb{Q})$ of \mathcal{E} . Let $\hat{\phi}^1, \dots, \hat{\phi}^M$ be the dual basis of $\hat{\phi}_1, \dots, \hat{\phi}_M$ with respect to the orbifold Poincaré pairing.

We will use X to denote the coarse moduli space of \mathcal{X} and use E to denote the coarse moduli space of \mathcal{E} . Then the \mathbb{C}^\times action on \mathcal{X} descends to the \mathbb{C}^\times action on X with E being the associated bundle. Following [8] and [5], there is a (non-canonical) splitting

$$H^*(\mathcal{E}; \mathbb{Q}) \cong H^*(E; \mathbb{Q}) \cong H^*(X; \mathbb{Q}) \otimes H^*(\mathbb{P}^1; \mathbb{Q}) \cong H^*(\mathcal{X}; \mathbb{Q}) \otimes H^*(\mathbb{P}^1; \mathbb{Q}).$$

According to [5], there is a unique \mathbb{C}^\times -fixed component $F_{\max} \subset X^{\mathbb{C}^\times}$ such that the normal bundle of F_{\max} has only negative \mathbb{C}^\times -weights. Let σ_0 be the section associated to a fixed point in F_{\max} . Following [5], there is a splitting defined by this maximal section.

$$(1) \quad H_2(\mathcal{E}; \mathbb{Z}) / \text{tors} \cong H_2(E; \mathbb{Z}) / \text{tors} \cong \mathbb{Z}[\sigma_0] \oplus (H_2(X, \mathbb{Z}) / \text{tors}) \cong \mathbb{Z}[\sigma_0] \oplus (H_2(\mathcal{X}, \mathbb{Z}) / \text{tors}).$$

Let $NE(X) \subset H_2(X; \mathbb{R})$ denote the Mori cone, i.e. the cone generated by effective curves and set

$$NE(X)_{\mathbb{Z}} := NE(X) \cap (H_2(X, \mathbb{Z}) / \text{tors}).$$

Then, by lemma 2.2 of [5], we have

$$(2) \quad NE(E)_{\mathbb{Z}} = \mathbb{Z}_{\geq 0}[\sigma_0] + NE(X)_{\mathbb{Z}}.$$

Let $H_2^{sec}(E; \mathbb{Z})$ be the affine subspace of $H_2(E, \mathbb{Z}) / \text{tors}$ which consists of the classes that project to the positive generator of $H_2(\mathbb{P}^1; \mathbb{Z})$, we set

$$NE(E)_{\mathbb{Z}}^{sec} := NE(E)_{\mathbb{Z}} \cap H_2^{sec}(E; \mathbb{Z}),$$

then we obtain

$$(3) \quad NE(E)_{\mathbb{Z}}^{sec} = [\sigma_0] + NE(X)_{\mathbb{Z}}.$$

We choose a nef integral basis $\{p_1, \dots, p_r\}$ of $H^2(\mathcal{X}; \mathbb{Q})$, then there are unique lifts of p_1, \dots, p_r in $H^2(\mathcal{E}; \mathbb{Q})$ which vanish on $[\sigma_0]$. By abuse of notation, we also denote these lifts as p_1, \dots, p_r , these lifts are also nef. Let p_0 be the pullback of the positive generator of $H^2(\mathbb{P}^1; \mathbb{Z})$ in $H^2(\mathcal{E}; \mathbb{Q})$. Therefore, $\{p_0, p_1, \dots, p_r\}$ is an integral basis of $H^2(\mathcal{E}; \mathbb{Q})$.

Let q_0, q_1, \dots, q_r be the Novikov variables of \mathcal{E} dual to p_0, p_1, \dots, p_r and q_1, \dots, q_r be the Novikov variables of \mathcal{X} dual to p_1, \dots, p_r . We denote the Novikov ring of \mathcal{X} and the Novikov ring of \mathcal{E} by

$$\Lambda_{\mathcal{X}} := \mathbb{Q}[[q_1, \dots, q_r]] \quad \text{and} \quad \Lambda_{\mathcal{E}} := \mathbb{Q}[[q_0, q_1, \dots, q_r]],$$

respectively. For each $d \in NE(X)_{\mathbb{Z}}$, we write

$$q^d := q_1^{\langle p_1, d \rangle} \dots q_r^{\langle p_r, d \rangle} \in \Lambda_{\mathcal{X}};$$

and for each $\beta \in NE(E)_{\mathbb{Z}}$, we write

$$q^{\beta} := q_0^{\langle p_0, \beta \rangle} q_1^{\langle p_1, \beta \rangle} \dots q_r^{\langle p_r, \beta \rangle} \in \Lambda_{\mathcal{E}}.$$

The τ -deformed orbifold quantum product is defined as follows:

$$(4) \quad \alpha \bullet_{\tau} \beta = \sum_{d \in NE(X)_{\mathbb{Z}}} \sum_{l \geq 0} \sum_{k=1}^N \frac{1}{l!} \langle \alpha, \beta, \tau, \dots, \tau, \phi_k \rangle_{0, l+3, d}^{\mathcal{X}} q^d \phi^k,$$

the associated quantum cohomology ring is denoted by

$$QH_{\tau}(\mathcal{X}) := (H(\mathcal{X}) \otimes_{\mathbb{Q}} \Lambda_{\mathcal{X}}, \bullet_{\tau}).$$

Definition 2.2. The Seidel element of \mathcal{X} is the class

$$(5) \quad S(\hat{\tau}) := \sum_{\alpha} \sum_{\beta \in NE(E)_{\mathbb{Z}}^{sec}} \sum_{l \geq 0} \frac{1}{l!} \langle \mathbf{1}, \hat{\tau}_{tw}, \dots, \hat{\tau}_{tw}, \iota_* \phi_{\alpha} \psi \rangle_{0, l+2, \beta}^{\mathcal{E}} q^{\alpha} e^{\langle \hat{\tau}_{0,2}, \beta \rangle},$$

in $QH_{\tau}(\mathcal{X}) \otimes_{\Lambda_{\mathcal{X}}} \Lambda_{\mathcal{E}}$. Here $\iota : \mathcal{X} \rightarrow \mathcal{E}$ is the inclusion of a fiber, and

$$\iota_* : H^*(\mathcal{X}; \mathbb{Q}) \rightarrow H^{*+2}(\mathcal{E}; \mathbb{Q})$$

is the Gysin map. Moreover,

$$e^{\langle \hat{\tau}_{0,2}, \beta \rangle} = q^{\beta} = q_0^{\langle p_0, \beta \rangle} \dots q_r^{\langle p_r, \beta \rangle},$$

where

$$\hat{\tau}_{0,2} = \sum_{a=0}^r p_a \log q_a \in H^2(\mathcal{E}) \quad \text{and} \quad \hat{\tau} = \hat{\tau}_{0,2} + \hat{\tau}_{tw} \in H_{orb}^{\leq 2}(\mathcal{E}).$$

The Seidel element can be factorized as

$$(6) \quad S(\hat{\tau}) = q_0 \tilde{S}(\hat{\tau}), \quad \text{with} \quad \tilde{S}(\hat{\tau}) \in QH_{\tau}(\mathcal{X}).$$

2.2. J-functions. We will explain the relation between the Seidel element and the J -function of the associated bundle \mathcal{E} .

Definition 2.3. The J -function of \mathcal{E} is the cohomology valued function

$$(7) \quad J_{\mathcal{E}}(\hat{\tau}, z) = e^{\hat{\tau}_{0,2}/z} \left(1 + \sum_{\alpha} \sum_{(\beta, l) \neq (0,0), \beta \in NE(E)_{\mathbb{Z}}} \frac{e^{\langle \hat{\tau}_{0,2}, \beta \rangle}}{l!} \langle \mathbf{1}, \hat{\tau}_{tw}, \dots, \hat{\tau}_{tw}, \frac{\hat{\phi}_{\alpha}}{z - \psi} \rangle_{0, l+2, \beta}^{\mathcal{E}} \hat{\phi}^{\alpha} \right),$$

where $\frac{\hat{\phi}_{\alpha}}{z - \psi} = \sum_{n \geq 0} z^{-1-n} \hat{\phi}_{\alpha} \psi^n$.

Note that when $n = 0$, we will have

$$\begin{aligned} \text{(i)} \quad & \sum_{\alpha} \langle \mathbf{1}, \hat{\tau}_{tw}, \dots, \hat{\tau}_{tw}, \hat{\phi}_{\alpha} \rangle_{0,l+2,\beta}^{\mathcal{E}} \hat{\phi}^{\alpha} = 0, \quad \text{for } (l, \beta) \neq (1, 0); \\ \text{(ii)} \quad & \sum_{\alpha} \langle \mathbf{1}, \hat{\tau}_{tw}, \dots, \hat{\tau}_{tw}, \hat{\phi}_{\alpha} \rangle_{0,l+2,\beta}^{\mathcal{E}} \hat{\phi}^{\alpha} = \hat{\tau}_{tw}, \quad \text{for } (l, \beta) = (1, 0). \end{aligned}$$

The J -function can be expanded in terms of powers of z^{-1} as follows:

$$(8) \quad J_{\mathcal{E}}(\hat{\tau}, z) = e^{\sum_{a=0}^r p_a \log q_a / z} \left(1 + z^{-1} \hat{\tau}_{tw} + z^{-2} \sum_{n=0}^{\infty} F_n(q_1, \dots, q_r; \hat{\tau}) q_0^n + O(z^{-3}) \right),$$

where

$$(9) \quad F_n(q_1, \dots, q_r; \hat{\tau}) = \sum_{\alpha=1}^M \sum_{d \in NE(X)_{\mathbb{Z}}} \sum_{l \geq 0} \frac{1}{l!} \langle \mathbf{1}, \hat{\tau}_{tw}, \dots, \hat{\tau}_{tw}, \hat{\phi}_{\alpha} \rangle_{0,l+2,d+n\sigma_0}^{\mathcal{E}} q^d \hat{\phi}^{\alpha}$$

Proposition 2.4. *The Seidel element corresponding to the \mathbb{C}^{\times} action on \mathcal{X} is given by*

$$(10) \quad S(\hat{\tau}) = \iota^* (F_1(q_1, \dots, q_r; \hat{\tau}) q_0).$$

Proof. The proof in here is identical to the proof given in proposition 2.5 of [5] for smooth projective varieties:

Using the duality identity

$$\sum_{\alpha=1}^M \hat{\phi}_{\alpha} \otimes \iota^* \hat{\phi}^{\alpha} = \sum_{\alpha=1}^N \iota_* \phi_{\alpha} \otimes \phi^{\alpha},$$

we can see that

$$\iota^* F_1(q_1, \dots, q_r; \hat{\tau}) = \sum_{\alpha=1}^N \sum_{d \in NE(X)_{\mathbb{Z}}} \sum_{l \geq 0} \frac{1}{l!} \langle \mathbf{1}, \hat{\tau}_{tw}, \dots, \hat{\tau}_{tw}, \iota_* \phi_{\alpha} \rangle_{0,l+2,d+\sigma_0}^{\mathcal{E}} q^d \phi^{\alpha}.$$

Hence, the conclusion follows, i.e.

$$S(\hat{\tau}) = \iota^* (F_1(q_1, \dots, q_r; \hat{\tau}) q_0).$$

□

3. SEIDEL ELEMENTS CORRESPONDING TO TORIC DIVISORS

3.1. A Review of Toric Deligne-Mumford stacks. In this section, we will define toric Deligne-Mumford stacks following the construction of [1] and [6].

A toric Deligne-Mumford stack is defined by a stacky fan $\Sigma = (\mathbf{N}, \Sigma, \beta)$, where \mathbf{N} is a finitely generated abelian group, $\Sigma \subset \mathbf{N}_{\mathbb{Q}} = \mathbf{N} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a rational simplicial fan, and $\beta : \mathbb{Z}^m \rightarrow \mathbf{N}$ is a homomorphism. We assume β has finite cokernel and the rank of \mathbf{N} is n . The canonical map $\mathbf{N} \rightarrow \mathbf{N}_{\mathbb{Q}}$ generates the 1-skeleton of the fan Σ . Let \bar{b}_i be the image of b_i under this canonical map, where b_i is the image under β of the standard basis of \mathbb{Z}^m . Let $\mathbb{L} \subset \mathbb{Z}^m$ be the kernel of β . Then the fan sequence is the following exact sequence

$$(11) \quad 0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^m \xrightarrow{\beta} \mathbf{N}.$$

Let $\beta^\vee : (\mathbb{Z}^*)^m \rightarrow \mathbb{L}^\vee$ be the Gale dual of β in [1], where $\mathbb{L}^\vee := H^1(\text{Cone}(\beta)^*)$ is an extension of $\mathbb{L}^* = \text{Hom}(\mathbb{L}, \mathbb{Z})$ by a torsion subgroup. The divisor sequence is the following exact sequence

$$(12) \quad 0 \longrightarrow \mathbf{N}^* \xrightarrow{\beta^*} (\mathbb{Z}^*)^m \xrightarrow{\beta^\vee} \mathbb{L}^\vee.$$

By applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ to the dual map β^\vee , we have a homomorphism

$$\alpha : G \rightarrow (\mathbb{C}^\times)^m, \quad \text{where } G := \text{Hom}_{\mathbb{Z}}(\mathbb{L}^\vee, \mathbb{C}^\times),$$

and we let G act on \mathbb{C}^m via this homomorphism.

The collection of anti-cones \mathcal{A} is defined as follows:

$$\mathcal{A} := \left\{ I : \sum_{i \notin I} \mathbb{R}_{\geq 0} \bar{b}_i \in \Sigma \right\}.$$

Let \mathcal{U} denote the open subset of \mathbb{C}^m defined by \mathcal{A} :

$$\mathcal{U} := \mathbb{C}^m \setminus \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I,$$

where

$$\mathbb{C}^I = \{(z_1, \dots, z_m) : z_i = 0 \text{ for } i \notin I\}.$$

Definition 3.1. Following [6], the toric Deligne-Mumford stack \mathcal{X} is defined as the quotient stack

$$\mathcal{X} := [\mathcal{U}/G].$$

Remark 3.2. The toric variety X associated to the fan Σ is the coarse moduli space of \mathcal{X} [1].

Definition 3.3 ([6]). Given a stacky fan $\Sigma = (\mathbf{N}, \Sigma, \beta)$, we define the set of box elements $\text{Box}(\Sigma)$ as follows

$$\text{Box}(\Sigma) =: \left\{ v \in \mathbf{N} : \bar{v} = \sum_{k \notin I} c_k \bar{b}_k \text{ for some } 0 \leq c_k < 1, I \in \mathcal{A} \right\}$$

We assume that Σ is complete, then the connected components of the inertia stack \mathcal{IX} are indexed by the elements of $\text{Box}(\Sigma)$ (see [1]). Moreover, given $v \in \text{Box}(\Sigma)$, the age of the corresponding connected component of \mathcal{IX} is defined by $\text{age}(v) := \sum_{k \notin I} c_k$.

The Picard group $\text{Pic}(\mathcal{X})$ of \mathcal{X} can be identified with the character group $\text{Hom}(G, \mathbb{C}^\times)$. Hence

$$(13) \quad \mathbb{L}^\vee = \text{Hom}(G, \mathbb{C}^\times) \cong \text{Pic}(\mathcal{X}) \cong H^2(\mathcal{X}; \mathbb{Z}).$$

We can also use the extended stacky fans introduced by Jiang [7] to define the toric Deligne-Mumford stacks. Given a stacky fan $\Sigma = (\mathbf{N}, \Sigma, \beta)$ and a finite set

$$S = \{s_1, \dots, s_l\} \subset \mathbf{N}_\Sigma := \{c \in \mathbf{N} : \bar{c} \in |\Sigma|\}.$$

The S -extended stacky fan is given by $(\mathbf{N}, \Sigma, \beta^S)$, where $\beta^S : \mathbb{Z}^{m+l} \rightarrow \mathbf{N}$ is defined by:

$$(14) \quad \beta^S(e_i) = \begin{cases} b_i & 1 \leq i \leq m; \\ s_{i-m} & m+1 \leq i \leq m+l. \end{cases}$$

Let \mathbb{L}^S be the kernel of $\beta^S : \mathbb{Z}^{m+l} \rightarrow \mathbf{N}$. Then we have the following S -extended fan sequence

$$(15) \quad 0 \longrightarrow \mathbb{L}^S \longrightarrow \mathbb{Z}^{m+l} \xrightarrow{\beta^S} \mathbf{N}.$$

By the Gale duality, we have the S -extended divisor sequence

$$(16) \quad 0 \longrightarrow \mathbf{N}^* \xrightarrow{\beta^*} (\mathbb{Z}^*)^{m+l} \xrightarrow{\beta^{S^\vee}} \mathbb{L}^{S^\vee},$$

where $\mathbb{L}^{S^\vee} := H^1(\text{Cone}(\beta^S)^*)$.

Assumption 3.4. *In the rest of the paper, we will assume the set*

$$\{v \in \text{Box}(\Sigma); \text{age}(v) \leq 1\} \cup \{b_1, \dots, b_m\}$$

generates \mathbf{N} over \mathbb{Z} . And we choose the set

$$S = \{s_1, \dots, s_l\} \subset \text{Box}(\Sigma)$$

such that the set $\{b_1, \dots, b_m, s_1, \dots, s_l\}$ generates \mathbf{N} over \mathbb{Z} and $\text{age}(s_j) \leq 1$ for $1 \leq j \leq l$.

Let D_i^S be the image of the standard basis of $(\mathbb{Z}^*)^{m+l}$ under the map β^{S^\vee} , then there is a canonical isomorphism

$$(17) \quad \mathbb{L}^{S^\vee} \otimes \mathbb{Q} \cong (\mathbb{L}^\vee \otimes \mathbb{Q}) \bigoplus_{i=m+1}^{m+l} \mathbb{Q} D_i^S,$$

which can be constructed as follows ([6]):

Since Σ is complete, for $m < j \leq m+l$, the box element s_{j-m} is contained in some cone in Σ . Namely,

$$s_{j-m} = \sum_{i \notin I_j^S} c_{ji} b_i \quad \text{in } \mathbf{N} \otimes \mathbb{Q}, \quad c_{ji} \geq 0, \quad \exists I_j^S \in \mathcal{A}^S,$$

where I_j^S is the "anticone" of the cone containing s_{j-m} .

By the S -extended fan sequence 15 tensored with \mathbb{Q} , we have the following short exact sequence

$$0 \longrightarrow \mathbb{L}^S \otimes \mathbb{Q} \longrightarrow \mathbb{Q}^{m+l} \xrightarrow{\beta^S} \mathbf{N} \otimes \mathbb{Q} \longrightarrow 0.$$

Hence, there exists a unique $D_j^{S^\vee} \in \mathbb{L}^S \otimes \mathbb{Q}$ such that

$$(18) \quad \langle D_i^S, D_j^{S^\vee} \rangle = \begin{cases} 1 & i = j; \\ -c_{ji} & i \notin I_j^S; \\ 0 & i \in I_j^S \setminus \{j\}. \end{cases}$$

These vectors $D_j^{S^\vee}$ define a decomposition

$$\mathbb{L}^{S^\vee} \otimes \mathbb{Q} = \text{Ker}((D_{m+1}^{S^\vee}, \dots, D_{m+l}^{S^\vee}) : \mathbb{L}^{S^\vee} \otimes \mathbb{Q} \rightarrow \mathbb{Q}^l) \oplus \bigoplus_{j=m+1}^{m+l} \mathbb{Q} D_j^S.$$

We identify the first factor $\text{Ker}(D_{m+1}^{S^\vee}, \dots, D_{m+l}^{S^\vee})$ with $\mathbb{L}^\vee \otimes \mathbb{Q}$. Via this decomposition, we can regard $H^2(\mathcal{X}, \mathbb{Q}) \cong \mathbb{L}^\vee \otimes \mathbb{Q}$ as a subspace of $\mathbb{L}^{S^\vee} \otimes \mathbb{Q}$.

Let D_i be the image of D_i^S in $\mathbb{L}^\vee \otimes \mathbb{Q}$ under this decomposition. Then

$$D_i = 0, \quad \text{for } m+1 \leq i \leq m+l.$$

Let \mathcal{A}^S be the collection of S -extended anti-cones, i.e.

$$\mathcal{A}^S := \left\{ I^S : \sum_{i \notin I^S} \mathbb{R}_{\geq 0} \overline{\beta^S(e_i)} \in \Sigma \right\}.$$

Note that

$$\{s_1, \dots, s_l\} \subset I^S, \quad \forall I^S \in \mathcal{A}^S.$$

By applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ to the S -extended dual map β^\vee , we have a homomorphism

$$\alpha^S : G^S \rightarrow (\mathbb{C}^\times)^{m+l}, \quad \text{where } G^S := \text{Hom}_{\mathbb{Z}}(\mathbb{L}^{S^\vee}, \mathbb{C}^\times).$$

We define \mathcal{U} to be the open subset of \mathbb{C}^{m+l} defined by \mathcal{A}^S :

$$\mathcal{U}^S := \mathbb{C}^{m+l} \setminus \bigcup_{I^S \notin \mathcal{A}^S} \mathbb{C}^{I^S} = \mathcal{U} \times (\mathbb{C}^\times)^l,$$

where

$$\mathbb{C}^{I^S} = \{(z_1, \dots, z_{m+l}) : z_i = 0 \text{ for } i \notin I^S\}.$$

Let G^S act on \mathcal{U}^S via α^S . Then we obtain the quotient stack $[\mathcal{U}^S/G^S]$. Jiang [7] showed that

$$[\mathcal{U}^S/G^S] \cong [\mathcal{U}/G] = \mathcal{X}.$$

3.2. Mirror theorem for toric stacks. In [3], Coates-Corti-Iritani-Tseng defined the S -extended I -function of a smooth toric Deligne-Mumford stack \mathcal{X} with projective coarse moduli space and proved that this I -function is a point of Givental's Lagrangian cone \mathcal{L} for the Gromov-Witten theory of \mathcal{X} . In this paper, we will only need this theorem for the weak Fano case. In this case, the mirror theorem will take a particularly simple form which can be stated as an equality of I -function and J -function via a change of variables, called mirror map.

To state the mirror theorem for weak Fano toric Deligne-Mumford stack, we need the following definitions.

We define the S -extended Kähler cone $C_{\mathcal{X}}^S$ as

$$C_{\mathcal{X}}^S := \bigcap_{I^S \in \mathcal{A}^S} \Sigma_{i \in I^S} \mathbb{R}_{>0} D_i^S$$

and the Kähler cone $C_{\mathcal{X}}$ as

$$C_{\mathcal{X}} := \bigcap_{I \in \mathcal{A}} \Sigma_{i \in I} \mathbb{R}_{>0} D_i.$$

Let p_1^S, \dots, p_{r+l}^S be an integral basis of \mathbb{L}^{S^\vee} , where $r = m - n$, such that p_i^S is in the closure $\text{cl}(C_{\mathcal{X}}^S)$ of the S -extended Kähler cone $C_{\mathcal{X}}^S$ for all $1 \leq i \leq r + l$ and $p_{r+1}^S, \dots, p_{r+l}^S$ are in $\sum_{i=m+1}^{m+l} \mathbb{R}_{\geq 0} D_i^S$. We denote the image of p_i^S in $\mathbb{L}^\vee \otimes \mathbb{R}$ by p_i , therefore p_1, \dots, p_r are nef and p_{r+1}, \dots, p_{r+l} are zero. We define a matrix (m_{ia}) by

$$D_i^S = \sum_{a=1}^{r+l} m_{ia} p_a^S, \quad m_{ia} \in \mathbb{Z}.$$

Then the class D_i of toric divisor is given by

$$D_i = \sum_{a=1}^r m_{ia} p_a.$$

Definition 3.5 ([6], Section 3.1.4). A toric Deligne-Mumford stack \mathcal{X} is called weak Fano if the first Chern class ρ satisfies

$$\rho = c_1(T\mathcal{X}) = \sum_{i=1}^m D_i \in \text{cl}(C_{\mathcal{X}}),$$

where $C_{\mathcal{X}}$ is the Kähler cone of \mathcal{X} .

We will need a slightly stronger condition:

$$\rho^S := D_1^S + \dots + D_{m+l}^S \in \text{cl}(C_{\mathcal{X}}^S),$$

where $C_{\mathcal{X}}^S$ is the S -extended Kähler cone. By lemma 3.3 of [6], we can see that $\rho^S \in \text{cl}(C_{\mathcal{X}}^S)$ implies $\rho \in \text{cl}(C_{\mathcal{X}})$. Moreover, under assumption 3.4, we will have

$$\rho^S \in \text{cl}(C_{\mathcal{X}}^S) \quad \text{if and only if} \quad \rho \in \text{cl}(C_{\mathcal{X}}).$$

For a real number r , let $\lceil r \rceil$, $\lfloor r \rfloor$ and $\{r\}$ be the ceiling, floor and fractional part of r respectively.

Definition 3.6. We define two subsets \mathbb{K} and \mathbb{K}_{eff} of $L^S \otimes \mathbb{Q}$ as follows:

$$\begin{aligned} \mathbb{K} &:= \{d \in L^S \otimes \mathbb{Q}; \{i \in \{1, \dots, m+l\}; \langle D_i^S, d \rangle \in \mathbb{Z}\} \in \mathcal{A}^S\}, \\ \mathbb{K}_{\text{eff}} &:= \{d \in L^S \otimes \mathbb{Q}; \{i \in \{1, \dots, m+l\}; \langle D_i^S, d \rangle \in \mathbb{Z}_{\geq 0}\} \in \mathcal{A}^S\}. \end{aligned}$$

Remark 3.7. We will use $\mathbb{K}_{\mathcal{E}_j}$ and $\mathbb{K}_{\text{eff}, \mathcal{E}_j}$ to denote the corresponding sets for the associated bundle \mathcal{E}_j , and use $\mathbb{K}_{\mathcal{X}}$ and $\mathbb{K}_{\text{eff}, \mathcal{X}}$ to denote the corresponding sets for \mathcal{X} .

Definition 3.8 ([6], Section 3.1.3). The reduction function v is defined as follows:

$$\begin{aligned} v : \mathbb{K} &\longrightarrow \text{Box}(\Sigma) \\ d &\longmapsto \sum_{i=1}^m \lceil \langle D_i^S, d \rangle \rceil b_i + \sum_{j=1}^l \lceil \langle D_{m+j}^S, d \rangle \rceil s_j \end{aligned}$$

By the S -extended fan exact sequence, we have

$$\sum_{i=1}^m \langle D_i^S, d \rangle b_i + \sum_{j=1}^l \langle D_{m+j}^S, d \rangle s_j = 0 \in \mathbf{N} \otimes \mathbb{Q}.$$

Moreover, by the definition of \mathbb{K} , we have

$$\langle D_{m+j}^S, d \rangle \in \mathbb{Z}, \quad \text{for all } d \in \mathbb{K} \quad \text{and} \quad 1 \leq j \leq l.$$

Hence,

$$v(d) = \sum_{i=1}^m \{-\langle D_i^S, d \rangle\} b_i + \sum_{j=1}^l \{-\langle D_{m+j}^S, d \rangle\} s_j = \sum_{i=1}^m \{-\langle D_i^S, d \rangle\} b_i.$$

By abuse of notation, we use D_i to denote the divisor $\{z_i = 0\} \subset \mathcal{X}$ and the cohomology class in $H^2(\mathcal{X}; \mathbb{Z}) \cong \mathbb{L}^\vee$, for $1 \leq i \leq m$.

We consider the \mathbb{C}^\times -action fixing a toric divisor D_j , $1 \leq j \leq m$, the action of \mathbb{C}^\times on \mathbb{C}^m is given by

$$(z_1, \dots, z_m) \mapsto (z_1, \dots, t^{-1} z_j, \dots, z_m), \quad t \in \mathbb{C}^\times.$$

We can extend this to the diagonal \mathbb{C}^\times -action on $\mathcal{U} \times (\mathbb{C}^2 \setminus \{0\})$ by

$$(z_1, \dots, z_m, u, v) \mapsto (z_1, \dots, t^{-1} z_j, \dots, z_m, tu, tv), \quad t \in \mathbb{C}^\times.$$

The associated bundle \mathcal{E}_j of the \mathbb{C}^\times -action on \mathcal{X} is given by

$$\mathcal{E}_j = \mathcal{U} \times (\mathbb{C}^2 \setminus \{0\}) / G \times \mathbb{C}^\times.$$

We can also use the S -extended stacky fan of \mathcal{X} to define \mathcal{E}_j :

$$\mathcal{E}_j = \mathcal{U}^S \times (\mathbb{C}^2 \setminus \{0\}) / G^S \times \mathbb{C}^\times.$$

Therefore \mathcal{E}_j is also a toric Deligne-Mumford stack. We can identify $H^2(\mathcal{E}_j; \mathbb{Z})$ with the lattice of the characters of $G \times \mathbb{C}^\times$:

$$(19) \quad H^2(\mathcal{E}_j; \mathbb{Z}) \cong \mathbb{L}^\vee \oplus \mathbb{Z} \cong H^2(\mathcal{X}; \mathbb{Z}) \oplus \mathbb{Z}.$$

Moreover, we have the divisor sequence

$$0 \rightarrow \mathbf{N}^* \oplus \mathbb{Z} \rightarrow (\mathbb{Z}^*)^{m+2} \rightarrow \mathbb{L}^\vee \oplus \mathbb{Z}.$$

And the S -extended divisor sequence

$$0 \rightarrow \mathbf{N}^* \oplus \mathbb{Z} \rightarrow (\mathbb{Z}^*)^{m+l+2} \rightarrow \mathbb{L}^{S^\vee} \oplus \mathbb{Z}.$$

Let \hat{D}_i^S be the image of the standard basis of $(\mathbb{Z}^*)^{m+l+2}$ in $\mathbb{L}^{S^\vee} \oplus \mathbb{Z}$. Then

$$(20) \quad \hat{D}_i^S = (D_i^S, 0), \text{ for } i \neq j; \quad \hat{D}_j^S = (D_j^S, -1); \quad \hat{D}_{m+l+1}^S = \hat{D}_{m+l+2}^S = (0, 1).$$

And,

$$(21) \quad \hat{D}_i = (D_i, 0), \text{ for } i \neq j; \quad \hat{D}_j = (D_j, -1); \quad \hat{D}_{m+1} = \hat{D}_{m+2} = (0, 1).$$

The fan Σ_j of \mathcal{E}_j is a rational simplicial fan contained in $N_{\mathbb{Q}} \oplus \mathbb{Q}$. The 1-skeleton is given by

$$(22) \quad \hat{b}_i = (b_i, 0), \text{ for } 1 \leq i \leq m; \quad \hat{b}_{m+1} = (0, 1); \quad \hat{b}_{m+2} = (b_j, -1).$$

We set

$$p_0 := (0, 1) = \hat{D}_{m+1} = \hat{D}_{m+2} \in H^2(\mathcal{E}_j; \mathbb{Q}),$$

then a nef integral basis $\{p_1, \dots, p_r\}$ of $H^2(\mathcal{X}; \mathbb{Q})$ can be lifted to a nef integral basis $\{p_0, p_1, \dots, p_r\}$ of $H^2(\mathcal{E}_j; \mathbb{Q})$, under the splitting (19). Let p_1^S, \dots, p_{r+l}^S be an integral basis of \mathbb{L}^{S^\vee} , such that p_i is the image of p_i^S in $\mathbb{L}^\vee \otimes \mathbb{R}$. Let $p_0^S, p_1^S, \dots, p_{r+l}^S$ be an integral basis of $\mathbb{L}^{S^\vee} \oplus \mathbb{Z}$ and p_0 is the image of

$$p_0^S = \hat{D}_{m+l+1}^S = \hat{D}_{m+l+2}^S$$

in $(\mathbb{L}^\vee \oplus \mathbb{Z}) \otimes \mathbb{R}$. Note that p_{r+1}, \dots, p_{r+l} are zero. We have

$$C_{\mathcal{E}_j}^S = C_{\mathcal{X}}^S + \mathbb{R}_{>0} p_0^S, \quad \rho_{\mathcal{E}_j}^S = \rho_{\mathcal{X}}^S + p_0^S.$$

The following result is straightforward.

Lemma 3.9. *If $\rho_{\mathcal{X}}^S \in \text{cl}(C_{\mathcal{X}}^S)$, then $\rho_{\mathcal{E}_j}^S \in \text{cl}(C_{\mathcal{E}_j}^S)$, for $1 \leq j \leq m$.*

Definition 3.10. The I -function of \mathcal{X} is the $H_{orb}^*(\mathcal{X})$ -valued function:

$$(23) \quad I_{\mathcal{X}}(y, z) = e^{\sum_{i=1}^r p_i \log y_i / z} \sum_{d \in \mathbb{K}_{\text{eff}, \mathcal{X}}} \prod_{i=1}^{m+l} \left(\frac{\prod_{k=\lceil \langle D_i^S, d \rangle \rceil}^{\infty} (D_i + (\langle D_i^S, d \rangle - k)z)}{\prod_{k=0}^{\infty} (D_i + (\langle D_i^S, d \rangle - k)z)} \right) y^{d \cdot \mathbf{1}_{v(d)}},$$

where $y^d = y_1^{\langle p_1^S, d \rangle} \cdots y_{r+l}^{\langle p_{r+l}^S, d \rangle}$. Similarly, The I -function of \mathcal{E} is the $H_{orb}^*(\mathcal{E})$ -valued function:

$$(24) \quad I_{\mathcal{E}_j}(y, z) = e^{\sum_{i=0}^r p_i \log y_i / z} \sum_{\beta \in \mathbb{K}_{\text{eff}, \mathcal{E}_j}} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta \rangle \rceil}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z)}{\prod_{k=0}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z)} \right) y^\beta \mathbf{1}_{v(\beta)},$$

where $y^\beta = y_0^{\langle p_0^S, \beta, \rangle} y_1^{\langle p_1^S, \beta \rangle} \cdots y_{r+l}^{\langle p_{r+l}^S, \beta \rangle}$.

Following section 4.1 of [6], The I -functions of \mathcal{X} and \mathcal{E}_j can be rewritten in the form:

$$(25) \quad I_{\mathcal{X}}(y, z) = e^{\sum_{i=1}^r p_i \log y_i / z} \sum_{d \in \mathbb{K}_{\mathcal{X}}} \prod_{i=1}^{m+l} \left(\frac{\prod_{k=\lceil \langle D_i^S, d \rangle \rceil}^{\infty} (D_i + (\langle D_i^S, d \rangle - k) z)}{\prod_{k=0}^{\infty} (D_i + (\langle D_i^S, d \rangle - k) z)} \right) y^d \mathbf{1}_{v(d)},$$

and

$$(26) \quad I_{\mathcal{E}_j}(y, z) = e^{\sum_{i=0}^r p_i \log y_i / z} \sum_{\beta \in \mathbb{K}_{\mathcal{E}_j}} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta \rangle \rceil}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z)}{\prod_{k=0}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z)} \right) y^\beta \mathbf{1}_{v(\beta)},$$

respectively, because the summand with $d \in \mathbb{K} \setminus \mathbb{K}_{\text{eff}}$ vanishes. We refer to [6] for more details.

Theorem 3.11 ([6], Conjecture 4.3). *Assume that $\rho^S \in \text{cl}(C_{\mathcal{X}}^S)$. Then the I -function and the J -function satisfy the following relation:*

$$(27) \quad I_{\mathcal{X}}(y, z) = J_{\mathcal{X}}(\tau(y), z)$$

where

$$(28) \quad \tau(y) = \tau_{0,2}(y) + \tau_{tw}(y) = \sum_{i=1}^r (\log y_i) p_i + \sum_{j=m+1}^{m+l} y^{D_j^{S^\vee}} \mathfrak{D}_j + \text{h.o.t.} \in H_{orb}^{\leq 2}(\mathcal{X}),$$

with

$$\begin{aligned} \tau_{0,2}(y) &\in H^2(\mathcal{X}), \quad \tau_{tw}(y) \in H_{orb}^{\leq 2}(\mathcal{X}) \setminus H^2(\mathcal{X}), \\ \mathfrak{D}_j &= \prod_{i \notin I_j} D_i^{\lfloor c_{ji} \rfloor} \mathbf{1}_{v(D_j^{S^\vee})} \in H_{orb}^*(\mathcal{X}). \end{aligned}$$

and h.o.t. stands for higher order terms in z^{-1} . Furthermore, $\tau(y)$ is called the mirror map and takes values in $H_{orb}^{\leq 2}(\mathcal{X})$.

For $\tau_{0,2}(y) = \sum_{a=1}^r p_a \log q_a \in H^2(\mathcal{X})$, we have

$$\log q_i = \log y_i + g_i(y_1, \dots, y_{r+l}), \text{ for } i = 1, \dots, r,$$

where g_i is a (fractional) power series in y_1, \dots, y_{r+l} which is homogeneous of degree zero with respect to the degree $\deg y^d = 2\langle \rho_{\mathcal{X}}^S, d \rangle$.

By lemma 3.9, under the assumption of theorem 3.11, we can also apply the mirror theorem to the associated bundle \mathcal{E}_j , hence we have

$$I_{\mathcal{E}_j}(y, z) = J_{\mathcal{E}_j}(\tau^{(j)}(y), z),$$

where

$$\tau^{(j)}(y) = \tau_{0,2}^{(j)} + \tau_{tw}^{(j)}(y) \in H^2(\mathcal{E}_j) \oplus \left(H_{orb}^{\leq 2}(\mathcal{E}_j) \setminus H^2(\mathcal{E}_j) \right)$$

Since $\tau_{0,2}^{(j)}(y) = \sum_{a=0}^r p_a \log q_a \in H^2(\mathcal{E}_j)$, therefore

$$\log q_i = \log y_i + g_i^{(j)}(y_0, \dots, y_{r+l}), \text{ for } i = 0, \dots, r,$$

where $g_i^{(j)}$ is a (fractional) power series in y_0, y_1, \dots, y_{r+l} which is homogeneous of degree zero with respect to the degree $\deg y^\beta = 2\langle \rho_{\mathcal{E}_j}^S, \beta \rangle$.

3.3. Seidel elements and mirror maps.

Proposition 3.12. *The function $g_i^{(j)}$ does not depend on y_0 and we have*

$$g_i^{(j)}(y_0, \dots, y_{r+l}) = g_i(y_1, \dots, y_{r+l}), \text{ for } i = 1, \dots, r.$$

Proof. The functions g_i is the coefficients of $z^{-1}p_i$ in the expansion of $I_{\mathcal{X}}$:

$$I_{\mathcal{X}}(y, z) = e^{\sum_{i=1}^r p_i \log y_i / z} \left(1 + z^{-1} \left(\sum_{i=1}^r g_i(y) p_i + \tau_{tw} \right) + O(z^{-2}) \right).$$

The functions $g_i^{(j)}$ is the coefficients of $z^{-1}p_i$ in the expansion of $I_{\mathcal{E}_j}$:

$$I_{\mathcal{E}_j}(y, z) = e^{\sum_{i=0}^r p_i \log y_i / z} \left(1 + z^{-1} \left(\sum_{i=0}^r g_i^{(j)}(y) p_i + \tau_{tw}^{(j)} \right) + O(z^{-2}) \right).$$

Following the proof of lemma 3.5 of [5], we obtain the conclusion of this proposition. \square

We will prove $\tau_{tw}^{(j)}$ is also independent from y_0 . To begin with, the following lemma implies that $\tau_{tw}^{(j)}(y)$ is an (integer) power series in y_0 .

Lemma 3.13. *For any $\beta \in \mathbb{K}_{\mathcal{E}_j}$, we have $\langle p_0^S, \beta \rangle \in \mathbb{Z}$. Furthermore, for any $\beta \in \mathbb{K}_{\text{eff}, \mathcal{E}_j}$, we have $\langle p_0^S, \beta \rangle \in \mathbb{Z}_{\geq 0}$.*

Proof. Any cone $\sigma \in \Sigma_j$ containing both \hat{b}_{m+1} and \hat{b}_{m+2} should also contain \hat{b}_j , this is impossible since the fan Σ_j is simplicial and \hat{b}_{m+1} , \hat{b}_{m+2} and \hat{b}_j lie in the same plane. Hence, by the definition of $\mathbb{K}_{\mathcal{E}_j}$ (resp. $\mathbb{K}_{\text{eff}, \mathcal{E}_j}$), at least one of $\langle \hat{D}_{m+1}^S, \beta \rangle$ and $\langle \hat{D}_{m+2}^S, \beta \rangle$ has to be integer (resp. non-negative integer), for any $\beta \in \mathbb{K}_{\mathcal{E}_j}$ (resp. $\beta \in \mathbb{K}_{\text{eff}, \mathcal{E}_j}$). On the other hand, we have,

$$\langle p_0^S, \beta \rangle = \langle \hat{D}_{m+1}^S, \beta \rangle = \langle \hat{D}_{m+2}^S, \beta \rangle.$$

Therefore, we must have $\langle p_0^S, \beta \rangle \in \mathbb{Z}$ (resp. $\langle p_0^S, \beta \rangle \in \mathbb{Z}_{\geq 0}$). \square

As a direct consequence of the above lemma, $\tau_{tw}^{(j)}(y)$ can only contain non-negative integer power of y_0 .

Proposition 3.14. *Let $\tau_{tw}^{(j)}(y) = \sum_{n=0}^{\infty} H_n^{(j)}(y) y_0^n$, where $H_n^{(j)}(y)$ is a (fractional) power series in y_1, \dots, y_n . Then*

$$H_n^{(j)}(y) = 0 \text{ for } n \geq 1,$$

i.e. $\tau_{tw}^{(j)}(y)$ is independent from y_0 . Moreover, we have

$$\tau_{tw}^{(j)}(y) = \tau_{tw}(y).$$

Proof. Recall $\tau_{tw}^{(j)}(y)$ is the coefficient of z^{-1} in (29)

$$e^{-\sum_{i=0}^r p_i \log y_i / z} I_{\mathcal{E}_j}(y, z) = \sum_{\beta \in \mathbb{K}_{\text{eff}}, \mathcal{E}_j} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta \rangle \rceil}^{\infty} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)}{\prod_{k=0}^{\infty} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)} \right) y^\beta \mathbf{1}_{v(\beta)},$$

valued in $H_{\text{orb}}^{\leq 2}(\mathcal{E}_j) \setminus H^2(\mathcal{E}_j)$. Hence, we only need to consider terms with $v(\beta) \neq 0$, or, equivalently, $v(d) \neq 0$, where d is the natural projection of β on to $\mathbb{K}_{\text{eff}, \mathcal{X}}$.

Therefore, it remains to examine the product factor:

$$\begin{aligned} & \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta \rangle \rceil}^{\infty} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)}{\prod_{k=0}^{\infty} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)} \right) \\ &= \frac{\prod_{i: \langle \hat{D}_i^S, \beta \rangle < 0} \prod_{\langle \hat{D}_i^S, \beta \rangle \leq k < 0} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)}{\prod_{i: \langle \hat{D}_i^S, \beta \rangle > 0} \prod_{0 \leq k < \langle \hat{D}_i^S, \beta \rangle} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)} \\ (30) \quad &= C_\beta z^{-\left(\sum_{i=1}^{m+l+2} \lceil \langle \hat{D}_i^S, \beta \rangle \rceil + \#\{i: \langle \hat{D}_i^S, \beta \rangle \in \mathbb{Z}_{<0}\} \right)} \prod_{i: \langle \hat{D}_i^S, \beta \rangle \in \mathbb{Z}_{<0}} \hat{D}_i + h.o.t., \end{aligned}$$

where

$$(31) \quad C_\beta = \prod_{i: \langle \hat{D}_i^S, \beta \rangle < 0} \prod_{\langle \hat{D}_i^S, \beta \rangle < k < 0} \left(\langle \hat{D}_i^S, \beta \rangle - k \right) \prod_{i: \langle \hat{D}_i^S, \beta \rangle > 0} \prod_{0 \leq k < \langle \hat{D}_i^S, \beta \rangle} \left(\langle \hat{D}_i^S, \beta \rangle - k \right)^{-1}.$$

By assumption, we need to have

$$\sum_{i=1}^{m+l+2} \lceil \langle \hat{D}_i^S, \beta \rangle \rceil \geq \sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle \geq 0.$$

The equality holds if and only if

$$\langle \hat{D}_i^S, \beta \rangle \in \mathbb{Z}, \quad \text{for all } 1 \leq i \leq m+l+2; \quad \text{and} \quad \sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle = 0.$$

However, this would imply $v(\beta) = 0$, hence we cannot have $\sum_{i=1}^{m+l+2} \lceil \langle \hat{D}_i^S, \beta \rangle \rceil = 0$. Therefore, the expansion (30) would contribute to $H_n^{(j)}$ only when

$$\sum_{i=1}^{m+l+2} \lceil \langle \hat{D}_i^S, \beta \rangle \rceil = 1 \quad \text{and} \quad \#\{i: \langle \hat{D}_i^S, \beta \rangle \in \mathbb{Z}_{<0}\} = 0.$$

In this case, if $\langle p_0^S, \beta \rangle \geq 1$, then

$$\sum_{i=1}^{m+l+2} \lceil \langle \hat{D}_i^S, \beta \rangle \rceil \geq \sum_{i=1}^{m+l} \lceil \langle D_i^S, d \rangle \rceil + 1,$$

therefore, we have

$$0 \geq \sum_{i=1}^{m+l} \lceil \langle D_i^S, d \rangle \rceil \geq \sum_{i=1}^{m+l} \langle D_i^S, d \rangle = 0.$$

This implies, when $\langle p_0^S, \beta \rangle \geq 1$, we must have

$$\langle D_i^S, d \rangle \in \mathbb{Z}, \quad \text{for } 1 \leq i \leq m+l.$$

It is a contradiction, since $\hat{\tau}_{tw} \in H_{orb}^{\leq 2}(\mathcal{E}_j) \setminus H^2(\mathcal{E}_j)$ implies $v(d) \neq 0$. Hence

$$H_n^{(j)} = 0 \text{ for all } n > 0$$

and $\tau_{tw}^{(j)}(y)$ is independent from y_0 . Moreover, by the expression of I -functions and the identity

$$\iota^* I_{\mathcal{E}_j} \big|_{y_0=0} = I_{\mathcal{X}},$$

we have $\tau_{tw}^{(j)}(y) = \tau_{tw}(y)$. □

As a direct consequence of the above lemma, we can use the following notation for the Seidel element

$$(32) \quad \tilde{S}_j(\tau(y)) := \tilde{S}_j(\tau^{(j)}(y)),$$

since $\tilde{S}_j(\tau^{(j)}(y))$ does not depend on y_0 or q_0 .

3.4. Seidel Elements in terms of I -functions. We can rewrite the I -function of the associated bundle \mathcal{E}_j as follows:

$$(33) \quad e^{\sum_{i=0}^r p_i \log y_i / z} \left(1 + z^{-1} \left(\sum_{i=0}^r g_i^{(j)}(y) p_i + \tau_{tw}^{(j)}(y) \right) + z^{-2} \left(\sum_{n=0}^2 G_n^{(j)}(y) y_0^n \right) + O(z^{-3}) \right).$$

Then, $\log q_i = \log y_i + g_i^{(j)}(y)$ implies

$$(34) \quad I_{\mathcal{E}_j}(y, z) = e^{\sum_{i=0}^r p_i \log q_i / z} \left(1 + z^{-1} \tau_{tw}^{(j)}(y) + z^{-2} \left(\sum_{n=0}^2 G_n^{(j)}(y) y_0^n \right) + O(z^{-3}) \right),$$

where $G_n^{(j)}(y)$ is a (fractional) power series in y_1, \dots, y_{r+l} taking values in $H_{orb}^*(\mathcal{E}_j)$.

By proposition (2.4), the Seidel element $\tilde{S}_j(\tau^{(j)}(y))$ is the coefficient of q_0/z^2 in

$$\exp \left(- \sum_{i=0}^r p_i \log q_i / z \right) J_{\mathcal{E}_j}(\tau^{(j)}(y), z),$$

hence $J_{\mathcal{E}_j}(\tau^{(j)}(y), z) = I_{\mathcal{E}_j}(y, z)$ and $\log q_0 = \log y_0 + g_0^{(j)}(y)$ imply the following result:

Theorem 3.15. *The Seidel element S_j associated to the toric divisor D_j is given by*

$$(35) \quad S_j(\tau^{(j)}(y)) = \iota^*(G_1^{(j)}(y) y_0).$$

Furthermore, we have

$$(36) \quad \tilde{S}_j(\tau(y)) = \tilde{S}_j(\tau^{(j)}(y)) = \exp(-g_0^{(j)}(y)) \iota^*(G_1^{(j)}(y)).$$

3.5. Computation of $g_0^{(j)}$. The computation is essentially the same as the proof of lemma 3.16 of [5]. Consider the product factors in $I_{\mathcal{E}_j}$:

$$\prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta \rangle \rceil}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k)z)}{\prod_{k=0}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k)z)} \right) y^{\beta} \mathbf{1}_{v(\beta)},$$

these factors contribute to $g_i^{(j)}$ if

$$v(\beta) = \sum_{i=1}^{m+l+2} \{-\langle \hat{D}_i^S, \beta \rangle\} \hat{b}_i = 0,$$

then, by the definition of \mathbb{K}_{eff} , we must have

$$\langle \hat{D}_i^S, \beta \rangle \in \mathbb{Z}, \text{ for all } 1 \leq i \leq m+l+2.$$

In this case, the product factors can be rewritten as

$$\begin{aligned} & \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta \rangle \rceil}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k)z)}{\prod_{k=0}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k)z)} \right) y^{\beta} \mathbf{1}_{v(\beta)} \\ &= \prod_{i=1}^{m+l+2} \frac{\prod_{k=-\infty}^0 (\hat{D}_i + kz)}{\prod_{k=-\infty}^{\langle \hat{D}_i^S, \beta \rangle} (\hat{D}_i + kz)} y^{\beta} \\ (37) \quad &= \left(C_{\beta} z^{-\sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle - \#\{i : \langle \hat{D}_i^S, \beta \rangle < 0\}} \prod_{i : \langle \hat{D}_i^S, \beta \rangle < 0} \hat{D}_i + h.o.t. \right) y^{\beta}, \end{aligned}$$

where *h.o.t.* stands for higher order terms in z^{-1} and

$$(38) \quad C_{\beta} = \prod_{i : \langle \hat{D}_i^S, \beta \rangle < 0} (-1)^{-\langle \hat{D}_i^S, \beta \rangle - 1} \left(-\langle \hat{D}_i^S, \beta \rangle - 1 \right)! \prod_{i : \langle \hat{D}_i^S, \beta \rangle \geq 0} \left(\langle \hat{D}_i^S, \beta \rangle! \right)^{-1}.$$

They contribute to the z^{-1} term if

$$\sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle + \#\{i : \langle \hat{D}_i^S, \beta \rangle < 0\} \leq 1.$$

Since we assume $\rho_{\mathcal{X}}^S \in cl(C_{\mathcal{X}}^S)$, hence $\rho_{\mathcal{E}_j}^S \in cl(C_{\mathcal{E}_j}^S)$. So it has to be the following three cases:

- $\begin{cases} \sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle = 0 \\ \#\{i : \langle \hat{D}_i^S, \beta \rangle \in \mathbb{Z}_{<0}\} = 0 \end{cases}$
- $\begin{cases} \sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle = 1 \\ \#\{i : \langle \hat{D}_i^S, \beta \rangle \in \mathbb{Z}_{<0}\} = 0 \end{cases}$
- $\begin{cases} \sum_{i=1}^{m+l+2} \lceil \langle \hat{D}_i^S, \beta \rangle \rceil = 0 \\ \#\{i : \langle \hat{D}_i^S, \beta \rangle \in \mathbb{Z}_{<0}\} = 1 \end{cases}.$

In the first case, we have $\langle \hat{D}_i^S, \beta \rangle = 0$ for all i , hence $\beta = 0$; the second case can not happen, since β has to satisfy $\langle \hat{D}_i^S, \beta \rangle = 0$ except for one i and this implies $\beta = 0$.

Therefore, the coefficient of z^{-1} is from the third case, where

$$(39) \quad \sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle = 0 \quad \text{and} \quad \#\{i : \langle \hat{D}_i^S, \beta \rangle < 0\} = 1.$$

By the assumption $\rho_{\mathcal{X}}^S \in cl(C_{\mathcal{X}}^S)$, we must have $\sum_{i=1}^{m+l} \langle D_i^S, d \rangle = 0$ and $\langle p_0^S, \beta \rangle = 0$. Moreover, $\langle D_i^S, d \rangle < 0$ for exactly one i in $\{1, \dots, m\}$. (Note that $\langle D_i^S, d \rangle \geq 0$ for $i \in \{m+1, \dots, m+l\}$.)

Now $g_0^{(j)}$ is the coefficient corresponding to p_0 and $\hat{D}_j = \langle D_j, -1 \rangle = D_j - p_0$ is the only one, among $\hat{D}_1, \dots, \hat{D}_m$, which contains p_0 . By expression (37), we must have $\langle D_j^S, d \rangle < 0$ and $\langle D_i^S, d \rangle \geq 0$ for $i \neq j$. Hence we have

Lemma 3.16. *The coefficient $g_0^{(j)}$ is given by*

$$(40) \quad g_0^{(j)}(y_1, \dots, y_{r+l}) = \sum_{\substack{\langle D_i^S, d \rangle \in \mathbb{Z}, 1 \leq i \leq m+l \\ \langle \rho_{\mathcal{X}}^S, d \rangle = 0 \\ \langle D_j^S, d \rangle < 0 \\ \langle D_i^S, d \rangle \geq 0, \forall i \neq j}} \frac{(-1)^{-\langle D_j^S, d \rangle} (-\langle D_j^S, d \rangle - 1)!}{\prod_{i \neq j} \langle D_i^S, d \rangle!} y^d.$$

4. BATYREV ELEMENTS

In this section, we will extend the definition of the Batyrev elements in [5] to toric Deligne-Mumford stacks and explore their relationships with the Seidel elements.

4.1. Batyrev Elements. Following [6], consider the mirror coordinates y_1, \dots, y_{r+l} of the toric Deligne-Mumford stacks \mathcal{X} with $\rho_{\mathcal{X}}^S \in cl(C_{\mathcal{X}}^S)$. Set $\mathbb{C}[y^{\pm}] = \mathbb{C}[y_1^{\pm}, \dots, y_{r+l}^{\pm}]$.

Definition 4.1. The Batyrev ring $B(\mathcal{X})$ of \mathcal{X} is a $\mathbb{C}[y^{\pm}]$ -algebra generated by the variables $\lambda_1, \dots, \lambda_{r+l}$ with the following two relations:

$$(41) \quad \begin{aligned} \text{(multiplicative):} \quad & y^d \prod_{i: \langle D_i^S, d \rangle < 0} \omega_i^{-\langle D_i^S, d \rangle} = \prod_{i: \langle D_i^S, d \rangle > 0} \omega_i^{\langle D_i^S, d \rangle}, \quad d \in \mathbb{L}^{\mathbb{S}}; \\ \text{(linear):} \quad & \omega_i = \sum_{a=1}^{r+l} m_{ai} \lambda_a, \end{aligned}$$

where ω_i is invertible in $B(\mathcal{X})$.

Definition 4.2. We define the element $\tilde{p}_i^S \in H_{orb}^{\leq 2}(\mathcal{X}) \otimes \mathbb{Q}[[y_1, \dots, y_{r+l}]]$ as

$$\tilde{p}_i^S = \frac{\partial \tau(y)}{\partial \log y_i}, \quad i = 1, \dots, r+l.$$

Recall that

$$D_j^S = \sum_{i=1}^{r+l} m_{ij} p_i^S, \quad \text{for } 1 \leq j \leq m+l,$$

Then, the Batyrev element associated to D_j^S is defined by

$$\tilde{D}_j^S = \sum_{i=1}^{r+l} m_{ij} \tilde{p}_i^S, \quad \text{for } 1 \leq j \leq m+l.$$

Proposition 4.3. *The Batyrev elements $\tilde{D}_1^S, \dots, \tilde{D}_{m+l}^S$ satisfy the multiplicative and linear Batyrev relations for $\omega_j = \tilde{D}_j^S$.*

Proof. We consider the differential operator $\mathcal{P}_d \in \mathbb{C}[z, y^\pm, zy(\partial/\partial y)]$ for $d \in \mathbb{L}^S$, introduced by Iritani in [6], section 4.2:

$$(42) \quad \mathcal{P}_d := y^d \prod_{i: \langle D_i^S, d \rangle < 0} \prod_{k=0}^{-\langle D_i^S, d \rangle - 1} (\mathcal{D}_i - kz) - \prod_{i: \langle D_i^S, d \rangle > 0} \prod_{k=0}^{\langle D_i^S, d \rangle - 1} (\mathcal{D}_i - kz),$$

where $\mathcal{D}_i := \sum_{j=1}^{r+l} m_{ij} zy_j \partial / \partial y_j$.

By [6] lemma 4.6, we have

$$\mathcal{P}_d I(y, z) = 0, \quad d \in \mathbb{L}^S.$$

Hence

$$0 = \mathcal{P}_d(z, y, zy\partial/\partial y) I(y, z) = \mathcal{P}_d(z, y, zy\partial/\partial y) J(\tau(y), z).$$

This implies that

$$\mathcal{P}_d(z, y, z\tau^*\nabla) \mathbf{1} = 0,$$

where $\tau^*\nabla_i := \nabla_{\tau_*(y_i(\partial/\partial y_i))}$. Since

$$\tau(y) = \sum_{i=1}^r p_i \log y_i + \tau_{tw}(y) \quad \text{and} \quad \nabla_{\tau_*(y_i(\partial/\partial y_i))} = \tau_*(y_i(\partial/\partial y_i)) + \frac{1}{z} y_i \frac{\partial \tau(y)}{\partial y_i} \circ_\tau,$$

by setting $z = 0$, we proved that the Batyrev elements satisfy the multiplicative relation.

It is straightforward from the definition that the Batyrev elements satisfy the linear relation. \square

Consider the I -function for the bundle \mathcal{E}_j associated to the toric divisor D_j^S , for $1 \leq j \leq m$.

$$I_{\mathcal{E}_j}(y, z) = e^{\sum_{i=0}^r p_i \log y_i / z} \sum_{\beta \in \mathbb{K}_{\mathcal{E}_j}} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta \rangle \rceil}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k)z)}{\prod_{k=0}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k)z)} \right) y^\beta \mathbf{1}_{v(\beta)},$$

where $y^\beta = y_0^{\langle p_0^S, \beta \rangle} y_1^{\langle p_1^S, \beta \rangle} \dots y_{r+l}^{\langle p_{r+l}^S, \beta \rangle}$. The following lemma is a generalization of lemma 3.11 in [5].

Lemma 4.4. *The I -function $I_{\mathcal{E}_j}$ of the bundle \mathcal{E}_j , associated to the toric divisor D_j^S , satisfies the following partial differential equation:*

$$(43) \quad z \frac{\partial}{\partial y_0} \left(y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j} = \left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) - y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j}$$

Proof. Consider the left hand side of the equation (43),

$$\begin{aligned} & z \frac{\partial}{\partial y_0} \left(y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j} \\ &= e^{\sum_{i=0}^r p_i \log y_i / z} \sum_{\beta \in \mathbb{K}_{\mathcal{E}_j}} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta \rangle \rceil}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k)z)}{\prod_{k=0}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k)z)} \right) (2p_0 \langle p_0^S, \beta \rangle + \langle p_0^S, \beta \rangle^2 z) (y^\beta / y_0) \mathbf{1}_{v(\beta)}, \end{aligned}$$

and the right hand side of the equation (43)

$$\begin{aligned} & \left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) - y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j} \\ &= e^{\sum_{i=0}^r p_i \log y_i / z} \sum_{\beta \in \mathbb{K}_{\mathcal{E}_j}} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta \rangle \rceil}^{\infty} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)}{\prod_{k=0}^{\infty} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)} \right) \left(\hat{D}_j / z + \langle \hat{D}_j^S, \beta \rangle \right) y^\beta \mathbf{1}_{v(\beta)}. \end{aligned}$$

It is suffice to prove the coefficients of $y^\beta \mathbf{1}_{v(\beta)}$ in them are the same, for all $\beta \in \mathbb{K}_{\mathcal{E}_j}$. Note that, we can rewrite the product factor

$$\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta \rangle \rceil}^{\infty} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)}{\prod_{k=0}^{\infty} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)} = \frac{\prod_{k \leq 0, \{k\} = \{\langle \hat{D}_i^S, \beta \rangle\}} \left(\hat{D}_i + kz \right)}{\prod_{k \leq \langle \hat{D}_i^S, \beta \rangle, \{k\} = \{\langle \hat{D}_i^S, \beta \rangle\}} \left(\hat{D}_i + kz \right)}.$$

Let $\beta' = \beta + [\sigma_0]$, hence we have

$$\langle \hat{D}_j^S, \beta' \rangle = \langle \hat{D}_j^S, \beta \rangle - 1; \quad \langle \hat{D}_i^S, \beta' \rangle = \langle \hat{D}_i^S, \beta \rangle \text{ for } 1 \leq i \leq m+l \text{ and } i \neq j;$$

$$\langle \hat{D}_{m+l+1}^S, \beta' \rangle = \langle \hat{D}_{m+l+1}^S, \beta \rangle + 1; \quad \langle \hat{D}_{m+l+2}^S, \beta' \rangle = \langle \hat{D}_{m+l+2}^S, \beta \rangle + 1.$$

Note that $\beta \in \mathbb{K}_{\mathcal{E}_j}$ if and only if $\beta' \in \mathbb{K}_{\mathcal{E}_j}$. Moreover,

$$(y^{\beta'} / y_0) \mathbf{1}_{v(\beta')} = y^\beta \mathbf{1}_{v(\beta)}.$$

Hence the coefficient of $y^\beta \mathbf{1}_{v(\beta)}$ in $z \frac{\partial}{\partial y_0} (y_0 \frac{\partial}{\partial y_0}) I_{\mathcal{E}_j}$ is

$$\begin{aligned} & e^{\sum_{i=0}^r p_i \log y_i / z} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta \rangle \rceil}^{\infty} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)}{\prod_{k=0}^{\infty} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)} \right) \frac{\hat{D}_j + \langle \hat{D}_j^S, \beta \rangle z}{(p_0 + (\langle p_0^S, \beta \rangle + 1) z)^2} \bullet \\ & \bullet (2p_0(\langle p_0^S, \beta \rangle + 1) + (\langle p_0^S, \beta \rangle + 1)^2 z) \\ &= e^{\sum_{i=0}^r p_i \log y_i / z} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta \rangle \rceil}^{\infty} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)}{\prod_{k=0}^{\infty} \left(\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k) z \right)} \right) \frac{\hat{D}_j + \langle \hat{D}_j^S, \beta \rangle z}{z} \quad (\text{since } p_0^2 = 0). \end{aligned}$$

This is exactly the coefficient of $y^\beta \mathbf{1}_{v(\beta)}$ in $\left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) - y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j}$,

Hence the I-function of \mathcal{E}_j satisfies the differential equation

$$z \frac{\partial}{\partial y_0} \left(y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j} = \left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) - y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j}.$$

□

Using the expansion of $I_{\mathcal{E}_j}$, we have

$$I_{\mathcal{E}_j}(y, z) = e^{\sum_{i=0}^r p_i \log y_i / z} \left(1 + z^{-1} \left(\sum_{i=0}^r g_i^{(j)}(y) p_i + \tau_{tw}^{(j)} \right) + z^{-2} \left(\sum_{n=0}^2 G_n^{(j)}(y) y_0^n \right) + O(z^{-3}) \right),$$

where $G_n^{(j)}$ is a (fractional) power series in y_1, \dots, y_{r+l} taking values in $H_{orb}^*(\mathcal{E}_j)$. Therefore, we obtain

$$\begin{aligned} y_0 \frac{\partial}{\partial y_0} I_{\mathcal{E}_j} &= \frac{p_0}{z} e^{\sum_{i=0}^r p_i \log y_i / z} \left(1 + z^{-1} \left(\sum_{i=0}^r g_i^{(j)}(y) p_i + \tau_{tw}^{(j)} \right) + z^{-2} \left(\sum_{n=0}^2 G_n^{(j)}(y) y_0^n \right) + O(z^{-3}) \right) \\ &\quad + e^{\sum_{i=0}^r p_i \log y_i / z} \left(z^{-2} \left(\sum_{n=1}^2 G_n^{(j)}(y) n y_0^n \right) + O(z^{-3}) \right). \end{aligned}$$

Therefore, the left hand side of equation (43) is

$$\begin{aligned} & z \frac{\partial}{\partial y_0} \left(y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j} \\ &= \frac{\partial}{\partial y_0} \left(p_0 e^{\sum_{i=0}^r p_i \log y_i / z} \left(1 + z^{-1} \left(\sum_{i=0}^r g_i^{(j)}(y) p_i + \tau_{tw}^{(j)} \right) + z^{-2} \left(\sum_{n=0}^2 G_n^{(j)}(y) y_0^n \right) + O(z^{-3}) \right) \right) \\ &\quad + \frac{\partial}{\partial y_0} \left(e^{\sum_{i=0}^r p_i \log y_i / z} \left(z^{-1} \left(\sum_{n=1}^2 G_n^{(j)}(y) n y_0^n \right) + O(z^{-2}) \right) \right) \\ &= p_0 e^{\sum_{i=0}^r p_i \log y_i / z} (O(z^{-2})) + \frac{p_0}{y_0 z} e^{\sum_{i=0}^r p_i \log y_i / z} \left(z^{-1} \left(\sum_{n=1}^2 G_n^{(j)}(y) n y_0^n \right) + O(z^{-2}) \right) \\ &\quad + e^{\sum_{i=0}^r p_i \log y_i / z} \left(z^{-1} \left(\sum_{n=1}^2 G_n^{(j)}(y) n^2 y_0^{n-1} + O(z^{-2}) \right) \right) \\ &= e^{\sum_{i=0}^r p_i \log y_i / z} \left(z^{-1} \left(\sum_{n=1}^2 G_n^{(j)}(y) n^2 y_0^{n-1} \right) + O(z^{-2}) \right). \end{aligned}$$

On the other hand, the pull-back of the right hand side of equation (43) by ι^* is

$$\begin{aligned} & \iota^* \left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) - y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j} \\ &= \left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) - y_0 \frac{\partial}{\partial y_0} \right) \iota^* I_{\mathcal{E}_j} \\ &= \left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) \right) (I_{\mathcal{X}} + O(y_0)) \\ &= z^{-1} \left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) \tau(y) \right) + O(z^{-2}) + O(y_0). \end{aligned}$$

Hence we conclude the following lemma.

Lemma 4.5. *The Batyrev element $\tilde{D}_j(y)$ is given by*

$$(44) \quad \tilde{D}_j(y) = \iota^* G_1^{(j)}(y), \quad \text{for } 1 \leq j \leq m+l.$$

Hence, the following theorem is a direct consequence of the above lemma and theorem 3.15.

Theorem 4.6. *The Seidel element \tilde{S}_j corresponding to the toric divisor D_j is given by*

$$(45) \quad \tilde{S}_j(\tau(y)) = \exp(-g_0^j(y)) \tilde{D}_j(y).$$

4.2. The computation of \tilde{D}_j . Using the expansion

$$\left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) \right) I_{\mathcal{X}} = e^{\sum_{i=1}^r p_i \log y_i / z} \left(z^{-1} \tilde{D}_j + O(z^{-2}) \right),$$

we see that \tilde{D}_j is the coefficient of z^{-1} in the expansion of

$$e^{-\sum_{i=1}^r p_i \log y_i / z} \left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) \right) I_{\mathcal{X}}.$$

And, by direct computation

$$\begin{aligned} & \left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) \right) I_{\mathcal{X}} = \\ & e^{\sum_{i=1}^r p_i \log y_i / z} \sum_{d \in \mathbb{K}_{\text{eff}, \mathcal{X}}} \prod_{i=1}^{m+l} \left(\frac{\prod_{k=\lceil \langle D_i^S, d \rangle \rceil}^{\infty} (D_i + (\langle D_i^S, d \rangle - k) z)}{\prod_{k=0}^{\infty} (D_i + (\langle D_i^S, d \rangle - k) z)} \right) \left(\frac{D_j}{z} + \langle D_j^S, d \rangle \right) y^d \mathbf{1}_{v(d)}. \end{aligned}$$

Hence, to compute the Batyrev element \tilde{D}_j , it remains to examine the expansion of the product factor

$$\frac{\prod_{k=\lceil \langle D_i^S, d \rangle \rceil}^{\infty} (D_i + (\langle D_i^S, d \rangle - k) z)}{\prod_{k=0}^{\infty} (D_i + (\langle D_i^S, d \rangle - k) z)} = C_d z^{-(\sum_{i=1}^{m+l} \lceil \langle D_i^S, d \rangle \rceil + \#\{i : \langle D_i^S, d \rangle \in \mathbb{Z}_{<0}\})} \prod_{i : \langle D_i^S, d \rangle \in \mathbb{Z}_{<0}} D_i + h.o.t.,$$

where

$$(46) \quad C_d = \prod_{i : \langle D_i^S, d \rangle < 0} \prod_{\langle D_i^S, d \rangle < k < 0} (\langle D_i^S, d \rangle - k) \prod_{i : \langle D_i^S, d \rangle > 0} \prod_{0 \leq k < \langle D_i^S, d \rangle} (\langle D_i^S, d \rangle - k)^{-1}$$

The summand indexed by $d \in \mathbb{K}_{\text{eff}, \mathcal{X}}$ contributes to the coefficient of z^{-1} if and only if

$$\sum_{i=1}^{m+l} \lceil \langle D_i^S, d \rangle \rceil + \#\{i : \langle D_i^S, d \rangle \in \mathbb{Z}_{<0}\} \leq 1.$$

It happens only in the following three cases:

- $\sum_{i=1}^{m+l} \lceil \langle D_i^S, d \rangle \rceil + \#\{i : \langle D_i^S, d \rangle \in \mathbb{Z}_{<0}\} = 0$
- $\begin{cases} \sum_{i=1}^{m+l} \lceil \langle D_i^S, d \rangle \rceil = 0 \\ \#\{i : \langle D_i^S, d \rangle \in \mathbb{Z}_{<0}\} = 1 \end{cases}$
- $\begin{cases} \sum_{i=1}^{m+l} \lceil \langle D_i^S, d \rangle \rceil = 1 \\ \#\{i : \langle D_i^S, d \rangle \in \mathbb{Z}_{<0}\} = 0 \end{cases}$.

The first case happens if and only if $d = 0$. If the second case happens, then

$$\sum_{i=1}^{m+l} \lceil \langle D_i^S, d \rangle \rceil = \sum_{i=1}^{m+l} \langle D_i^S, d \rangle = \langle \rho_{\mathcal{X}}^S, d \rangle = 0.$$

In particular,

$$\langle D_i^S, d \rangle \in \mathbb{Z}, 1 \leq i \leq m+l.$$

Hence we obtain the following lemma:

Lemma 4.7. *For $1 \leq j \leq m+l$, the Batyrev element \tilde{D}_j is given by*
(47)

$$\tilde{D}_j = D_j + \sum_{i=1}^m D_i \sum_{\substack{\langle \rho_X^S, d \rangle = 0 \\ \langle D_i^S, d \rangle \in \mathbb{Z}_{<0} \\ \langle D_k^S, d \rangle \in \mathbb{Z}_{\geq 0}, \forall k \neq i}} C_d \langle D_j^S, d \rangle y^d + \sum_{\substack{\sum_{i=1}^{m+l} [\langle D_i^S, d \rangle] = 1 \\ \langle D_i^S, d \rangle \notin \mathbb{Z}_{<0}, \forall i}} C_d \langle D_j^S, d \rangle y^d \mathbf{1}_{v(d)},$$

where C_d is given by equation (46).

5. SEIDEL ELEMENTS CORRESPONDING TO BOX ELEMENTS

Consider the box element $s_j \in \text{Box}(\Sigma)$, such that

$$\bar{s}_j = \sum_{i=1}^m c_{ji} \bar{b}_i \in \mathbf{N}_{\mathbb{Q}}, \quad \text{for some } 0 \leq c_{ji} < 1.$$

Let n_j be the least common denominator of $\{c_{ji}\}_{i=1}^m$, we define a \mathbb{C}^\times -action on $\mathcal{U}^S \times (\mathbb{C}^2 \setminus \{0\})$ by

$$(z_1, \dots, z_{m+l}, u, v) \mapsto (t^{-c_{j1}n_j} z_1, \dots, t^{-c_{jm}n_j} z_m, z_{m+1}, \dots, z_{m+l}, t^{n_j} u, t^{n_j} v), \quad t \in \mathbb{C}^\times.$$

Hence we have an associated bundle

$$\mathcal{E}_{m+j} = \mathcal{U}^S \times (\mathbb{C}^2 \setminus \{0\}) / G^S \times \mathbb{C}^\times$$

over $\mathbb{CP}^1 \times B\mu_{n_j}$ with \mathcal{X} being the fiber. Furthermore, \mathcal{E}_{m+j} can also be considered as a bundle over \mathbb{CP}^1 , since there is a natural projection

$$\mathbb{CP}^1 \times B\mu_{n_j} \rightarrow \mathbb{CP}^1.$$

We can identify $H^2(\mathcal{E}_{m+j}; \mathbb{Z})$ with $H^2(\mathcal{X}; \mathbb{Z}) \oplus \mathbb{Z}$, where the second summand

$$\mathbb{Z} \cong \text{Pic}(\mathbb{CP}^1 \times B\mu_{n_j}),$$

and we have the following short exact sequence from remark 5.5 of [4]:

$$(48) \quad 0 \longrightarrow \text{Pic}(\mathbb{CP}^1) \longrightarrow \text{Pic}(\mathbb{CP}^1 \times B\mu_{n_j}) \longrightarrow \mathbb{Z}/n_j\mathbb{Z} \longrightarrow 0$$

We identify an element of $\text{Pic}(\mathbb{CP}^1)$ with its image in $\text{Pic}(\mathbb{CP}^1 \times B\mu_{n_j})$ under the above map. Then the weights of $G^S \times \mathbb{C}^\times$ defining \mathcal{E}_{m+j} are given by

$$\hat{D}_i^S = (D_i^S, -c_{ji}n_j), \quad \text{for } 1 \leq i \leq m; \quad \hat{D}_{m+j}^S = (D_{m+j}^S, 0) \quad \text{for } 1 \leq j \leq l;$$

$$\hat{D}_{m+l+1}^S = \hat{D}_{m+l+2}^S = (0, n_j).$$

The fan of \mathcal{E}_{m+j} is contained in $N_{\mathbb{Q}} \oplus \mathbb{Q}$. The 1-skeleton is given by

$$(49) \quad \hat{b}_i = (b_i, 0), \text{ for } 1 \leq i \leq m; \quad \hat{b}_{m+1} = (0, 1); \quad \hat{b}_{m+2} = (s_j, -1).$$

Let E_{m+j} be the coarse moduli space of \mathcal{E}_{m+j} . Then E_{m+j} is an X -bundle over \mathbb{CP}^1 . The Seidel element is defined as in equation (5).

We set

$$p_0 := (0, 1) \in H^2(E_{m+j}) \cong H^2(X) \oplus \text{Pic}(\mathbb{CP}^1),$$

a nef integral basis $\{p_1, \dots, p_r\}$ of $H^2(X; \mathbb{Q})$ can be lifted to a nef integral basis $\{p_0, p_1, \dots, p_r\}$ of $H^2(E_{m+j}; \mathbb{Q})$ such that the lift of p_i vanishes on the section class $[\sigma_0]$. There is an isomorphism between $H^2(E_{m+j}; \mathbb{Q})$ and $H^2(\mathcal{E}_{m+j}; \mathbb{Q})$, by abuse of notation, we identify p_i with its image in $H^2(\mathcal{E}_{m+j}; \mathbb{Q})$, for $0 \leq i \leq r$. Let p_1^S, \dots, p_{r+l}^S be an integral basis of \mathbb{L}^{S^\vee} , such that p_i is the image of p_i^S in $\mathbb{L}^\vee \otimes \mathbb{Q}$, under the canonical splitting of (17). Let $p_0^S, p_1^S, \dots, p_{r+l}^S$ be an integral basis of $\mathbb{L}^{S^\vee} \oplus \mathbb{Z}$ and p_0 be the image of

$$p_0^S = \hat{D}_{m+l+1}^S = \hat{D}_{m+l+2}^S$$

in $(\mathbb{L}^\vee \oplus \mathbb{Z}) \otimes \mathbb{R}$. Therefore p_{r+1}, \dots, p_{r+l} are zero.

As in the toric divisor case, we have the following expansion of the I -function:

(50)

$$\mathcal{I}_{\mathcal{E}_{m+j}}(y, z) = \sum_{i=0}^r p_i \log y_i / z \left(1 + z^{-1} \left(\sum_{i=0}^r g_i^{(m+j)}(y) p_i + \tau_{tw}^{(m+j)}(y) \right) + z^{-2} \left(\sum_{n=0}^2 G_n^{(m+j)}(y) y_0^n \right) + O(z^{-3}) \right),$$

and use the same argument as in lemma 3.12 and lemma 3.14, we can show that $g_i^{(m+j)}(y)$ and $\tau_{tw}^{(m+j)}(y)$ are independent from y_0 , for $1 \leq i \leq r$ and $1 \leq j \leq l$. Moreover, for each $j \in \{1, \dots, l\}$, we have

$$g_i^{(m+j)}(y_0, \dots, y_{r+l}) = g_i(y_1, \dots, y_{r+l}) \quad \text{for } i = 1, \dots, r.$$

And

$$\tau_{tw}^{(m+j)}(y) = \tau_{tw}(y).$$

We will also obtain the following theorem.

Theorem 5.1. *The Seidel element \tilde{S}_{m+j} associated to the box element s_j is given by*

$$(51) \quad \tilde{S}_{m+j}(\tau(y)) := \tilde{S}_{m+j}(\tau^{(m+j)}(y)) = \exp \left(-g_0^{(m+j)}(y) \right) \iota^* (G_1^{(m+j)}(y)).$$

Using the same computation as in the toric divisor case, we can compute the correction coefficient $g_0^{(m+j)}$:

Lemma 5.2. *The function $g_0^{(m+j)}$ is given by*

$$(52) \quad g_0^{(m+j)}(y_1, \dots, y_{r+l}) = \sum_{1 \leq k \leq m, k \notin I_j^S} \sum_{\substack{\langle D_i^S, d \rangle \in \mathbb{Z}, 1 \leq i \leq m+l \\ \langle \rho_{X_j}^S, d \rangle = 0 \\ \langle D_k^S, d \rangle < 0 \\ \langle D_i^S, d \rangle \geq 0, \forall i \neq k}} c_{jk} \frac{(-1)^{-\langle D_k^S, d \rangle} (-\langle D_k^S, d \rangle - 1)!}{\prod_{i \neq k} \langle D_i^S, d \rangle!} y^d,$$

where I_j^S is the "anticone" of the cone containing s_j .

Proof. The argument is almost the same as the argument in section 3.5. The only change we need to make is the paragraph above lemma 3.16:

In this case, $g_0^{(m+j)}$ is the coefficient corresponding to p_0 and elements in $\{\hat{D}_1, \dots, \hat{D}_m\}$ that contain p_0 are precisely these elements:

$$\hat{D}_k = \langle D_k, -c_{jk} n_j \rangle = D_k - c_{jk} p_0, \quad \text{for } 1 \leq k \leq m \quad \text{and} \quad k \notin I_j^S.$$

Therefore, by expression (37) and (39), we must have $\langle D_k^S, d \rangle < 0$ for exactly one k in $\{k \in \mathbb{Z} | 1 \leq k \leq m \text{ and } k \notin I_j^S\}$. \square

Moreover, by mimicking the computation in lemma 4.4, we have

Lemma 5.3. *the I -function of \mathcal{E}_{m+j} satisfies the following differential equation:*

$$(53) \quad z \frac{\partial}{\partial y_0} \left(y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j} = y^{-D_{m+j}^{\text{SV}}} \left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) - y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j},$$

where $D_{m+j}^{\text{SV}} \in \mathbb{L}^S \otimes \mathbb{Q}$ is defined by (18).

Proof. The proof is almost identical to the proof of lemma 4.4, except, this time, we will need to choose $\beta' = \beta + [\sigma_0] - D_{m+j}^{\text{SV}}$. Then everything else follows. \square

Using this lemma, following the argument in the toric divisor case, we conclude

Theorem 5.4. *The Seidel element \tilde{S}_{m+j} corresponding to the box element s_j , with*

$$\bar{s}_j = \sum_{i=1}^m c_{ji} \bar{b}_i, \quad \text{for some } 0 \leq c_{ji} < 1,$$

is given by

$$(54) \quad \tilde{S}_{m+j}(\tau^{(m+j)}(y)) = \exp \left(-g_0^{(m+j)} \right) y^{-D_{m+j}^{\text{SV}}} \tilde{D}_{m+j}(y),$$

where $\tilde{D}_{m+j}(y)$ is the corresponding Batyrev element. Moreover,

$$(55) \quad \tilde{D}_{m+j} = \sum_{i=1}^m D_i \sum_{\substack{\langle \rho_X^S, d \rangle = 0 \\ \langle D_i^S, d \rangle \in \mathbb{Z}_{<0} \\ \langle D_k^S, d \rangle \in \mathbb{Z}_{\geq 0}, \forall k \neq i}} C_d \langle D_{m+j}^S, d \rangle y^d + \sum_{\substack{\sum_{i=1}^{m+l} [\langle D_i^S, d \rangle] = 1 \\ \langle D_i^S, d \rangle \notin \mathbb{Z}_{<0}, \forall i}} C_d \langle D_{m+j}^S, d \rangle y^d \mathbf{1}_{v(d)},$$

and

$$(56) \quad C_d = \prod_{i: \langle D_i^S, d \rangle < 0} \prod_{\langle D_i^S, d \rangle < k < 0} (\langle D_i^S, d \rangle - k) \prod_{i: \langle D_i^S, d \rangle > 0} \prod_{0 \leq k < \langle D_i^S, d \rangle} (\langle D_i^S, d \rangle - k)^{-1}.$$

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